

# Unfriendly or weakly unfriendly partitions of graphs

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**Abstract.** For each infinite cardinal  $\kappa$  and each graph  $G = (V, E)$ , we say that a partition  $\pi : V \rightarrow \{0, 1\}$  is  $\kappa$ -unfriendly if, for each  $x \in V$ ,  $|\{y \in V \mid \{x, y\} \in E \text{ and } \pi(y) \neq \pi(x)\}| \geq |\{y \in V \mid \{x, y\} \in E \text{ and } \pi(y) = \pi(x)\}|$  or  $\geq \kappa$ ;  $\pi$  is unfriendly if the first property is true for each  $x \in V$ . Some uncountable graphs of infinite minimum degree without unfriendly partition have been constructed by S. Shelah and E.C. Milner, but it is not presently known if countable graphs without unfriendly partition exist.

We show that, for each integer  $n$ , each graph of infinite minimum degree has an  $\omega_n$ -unfriendly partition. We also prove that the following properties are equivalent: (i) each graph has an  $\omega$ -unfriendly partition; (ii) each countable graph has an unfriendly partition; (iii) each countable graph without nonempty induced subgraph of infinite minimum degree has an unfriendly partition (actually it is enough to consider a smaller class of graphs).

Here, a *graph* is a pair  $G = (V, E)$ , where  $V$  is the set of *vertices* and  $E$  is the set of *edges* of  $G$ ; the *edges* are non oriented pairs  $\{x, y\}$  with  $x, y \in V$  and  $x \neq y$ . We call *induced graphs* the graphs  $H = (W, F)$  with  $W \subset V$  and  $F = \{\{x, y\} \in E \mid x, y \in W\}$ .

The *neighbours* of a vertex  $x$  are the vertices  $y$  such that  $\{x, y\} \in E$ . The *degree* of  $x$  is the cardinal of its set of neighbours. The *minimum degree* of  $G$  is the minimum of the degrees of the vertices of  $G$ .

A *partition* of  $\Gamma$  is a map  $\pi : V \rightarrow \{0, 1\}$ . We say that  $\pi$  is *unfriendly* if, for each  $x \in V$ ,  $|\{y \in V \mid \{x, y\} \in E \text{ and } \pi(y) \neq \pi(x)\}| \geq |\{y \in V \mid \{x, y\} \in E \text{ and } \pi(y) = \pi(x)\}|$  (the two sets can be infinite). For each cardinal  $\kappa$ , we say that  $\pi$  is  $\kappa$ -*unfriendly* if, for each  $x \in V$ ,  $|\{y \in V \mid \{x, y\} \in E \text{ and } \pi(y) \neq \pi(x)\}| \geq \kappa$  or  $|\{y \in V \mid \{x, y\} \in E \text{ and } \pi(y) = \pi(x)\}| \geq \kappa$ .

## 1. Graphs of infinite minimum degree.

Here we consider the following question: For which cardinals  $\kappa$  is it true that any graph of infinite minimum degree has a  $\kappa$ -unfriendly partition?

**Theorem 1.1.** Let  $\kappa$  be an infinite cardinal and let  $G = (V, E)$  be a graph of infinite minimum degree. Suppose that each induced graph  $H = (W, F)$  with  $|W| \leq \kappa$  has an unfriendly partition if each element of  $W$  has  $\kappa$  neighbours

in  $W$ , or more neighbours in  $W$  than in  $V - W$ . Then  $G$  has a  $\kappa$ -unfriendly partition.

**Proof.** Let  $\mathcal{E}$  consist of the pairs  $(W, \pi)$ , with  $W \subset V$  and  $\pi : W \rightarrow \{0, 1\}$ , such that:

each element of  $W$  has at least  $\kappa$  neighbours in  $W$  or at least as many neighbours in  $W$  as in  $V - W$ ;

$\pi$  is a  $\kappa$ -unfriendly partition of the induced graph defined on  $W$ .

Then  $\mathcal{E}$  contains  $(\emptyset, \rho)$  where  $\rho : \emptyset \rightarrow \{0, 1\}$  is the trivial map. For each cardinal  $\mu$ , the union of any increasing sequence  $(W_\alpha, \pi_\alpha)_{\alpha < \mu}$  of elements of  $\mathcal{E}$  belongs to  $\mathcal{E}$ .

Consequently, it suffices to prove that, for each  $(W, \pi) \in \mathcal{E}$  with  $W \subsetneq V$ , there exists  $(W', \pi') \in \mathcal{E}$  with  $W \subsetneq W'$ .

First suppose that there exists  $x \in V - W$  with at least  $\kappa$  neighbours in  $W$ , or at least as many neighbours in  $W$  as in  $V - W$ . Then we write  $W' = W \cup \{x\}$  and  $\pi'(x) = 1$  except if

$$|\{y \in W \mid \{x, y\} \in E \text{ and } \pi(y) = 1\}| > |\{y \in W \mid \{x, y\} \in E \text{ and } \pi(y) = 0\}|.$$

Now we can suppose that each  $y \in V - W$  has less than  $\kappa$  neighbours in  $W$  and more neighbours in  $V - W$  than in  $W$ . Then we consider a vertex  $x \in V - W$  and we define by induction an increasing sequence  $(A_n)_{n \in \mathbb{N}}$  of subsets of  $V - W$ . We write  $A_0 = \{x\}$ . For each  $n \in \mathbb{N}$ ,  $A_{n+1}$  is obtained from  $A_n$  by adding, for each  $y \in A_n$ :

the neighbours of  $y$  in  $V - W$  if there exist at most  $\kappa$  of them;

$\kappa$  neighbours of  $y$  in  $V - W$  if there exist more than  $\kappa$  of them.

We write  $A = \cup_{n \in \mathbb{N}} A_n$ . We have  $|A| \leq \kappa$ . For each  $n \in \mathbb{N}$  and each  $y \in A_n$ , if  $y$  has at least  $\kappa$  neighbours in  $V$ , and therefore in  $V - W$ , then it has  $\kappa$  neighbours in  $A_{n+1}$  and therefore in  $A$ . Otherwise,  $y$  has more neighbours in  $A_{n+1}$  than in  $V - A_{n+1}$  since it has more neighbours in  $V - W$  than in  $W$ ; consequently,  $y$  has more neighbours in  $A$  than in  $V - A$ .

Now, according to the hypotheses of the Theorem, the induced graph defined on  $A$  admits an unfriendly partition  $\tau$ . It follows that  $\pi' = \tau \cup \pi$  is a  $\kappa$ -unfriendly partition of the induced graph defined on  $A \cup W$ . ■

By [2, Theorem 1], there exists a graph  $G = (V, E)$  of minimum degree  $\omega$  with  $|V| = (2^\omega)^{(+\omega)}$  which has no unfriendly partition. Moreover, by [2, Theorem 3], it is constant that there exists a graph  $G = (V, E)$  of minimum degree  $\omega$  with  $|V| = \omega_\omega$  which has no unfriendly partition. Consequently, in the case of graphs of infinite minimum degree, the following result is the best possible one:

**Corollary 1.2.** For each  $n \in \mathbb{N}$ , each graph of infinite minimum degree admits an  $\omega_n$ -unfriendly partition.

**Proof.** It follows from [1, Theorem 2] that, for each  $n \in \mathbb{N}$ , each graph of cardinal  $\omega_n$  with infinite minimum degree admits an unfriendly partition. The Corollary is a consequence of Theorem 1.1 and this fact. ■

## 2. Graphs of arbitrary minimum degree.

Now we consider the following question: Does each graph  $G = (V, E)$  admit an  $\omega$ -unfriendly partition? The answer is not presently known, even for countable graphs. We show that a positive answer for a particular class of countable graphs implies a positive answer for all graphs.

**Definitions.** A *graph with finite conditions* (resp. *with conditions*) is a pair  $\Gamma = (G, \mathcal{C})$  where  $G = (V, E)$  is a graph and  $\mathcal{C} = (\kappa_x, \lambda_x)_{x \in V}$  with  $\kappa_x, \lambda_x$  finite (resp. finite or infinite) cardinals for each  $x \in V$ .

We say that a map  $\pi : V \rightarrow \{0, 1\}$  is an *unfriendly partition* of  $\Gamma$  if, for each  $x \in V$ ,  $\pi(x) = 0$  implies

$$\kappa_x + |\{y \in V \mid \{x, y\} \in E \text{ and } \pi(y) = 0\}| \leq \lambda_x + |\{y \in V \mid \{x, y\} \in E \text{ and } \pi(y) = 1\}|$$

and  $\pi(x) = 1$  implies

$$\kappa_x + |\{y \in V \mid \{x, y\} \in E \text{ and } \pi(y) = 0\}| \geq \lambda_x + |\{y \in V \mid \{x, y\} \in E \text{ and } \pi(y) = 1\}|.$$

For each cardinal  $\mu$ , we say that  $\pi$  is a  $\mu$ -*unfriendly partition* of  $\Gamma$  if the two implications above are true, except possibly when the two cardinals considered are  $\geq \mu$ .

**Proposition 2.1.** Let  $\Gamma = (G, \mathcal{C})$  be a graph with conditions. Then there exists a set  $\mathcal{C}'$  of finite conditions, defined on  $G$  and trivial on the elements of infinite degree, such that each  $\omega$ -unfriendly partition of  $\Gamma' = (G, \mathcal{C}')$  is an  $\omega$ -unfriendly partition of  $\Gamma$ .

**Proof.** We write  $G = (V, E)$  and  $\mathcal{C} = (\kappa_x, \lambda_x)_{x \in V}$ . The set  $\mathcal{C}' = (\kappa'_x, \lambda'_x)_{x \in V}$  is defined as follows:

- $\kappa'_x = \lambda'_x = 0$  for  $x$  of infinite degree;
- $\kappa'_x = \kappa_x$  and  $\lambda'_x = \lambda_x$  for  $x$  of finite degree and  $\kappa_x, \lambda_x$  finite;
- $\kappa'_x = n + 1$  and  $\lambda'_x = 0$  for  $x$  of finite degree  $n$  and  $\kappa_x$  infinite;
- $\kappa'_x = 0$  and  $\lambda'_x = n + 1$  for  $x$  of finite degree  $n$ ,  $\kappa_x$  finite and  $\lambda_x$  infinite. ■

**Notation.** For each graph  $G = (V, E)$ , we denote by  $\mathcal{I}(G)$  the largest  $W \subset V$  such that each element of  $W$  has infinitely many neighbours in  $W$ .

**Remark.** We have  $\mathcal{I}(G) = V$  if and only if the minimum degree of  $G$  is infinite. We can have  $\mathcal{I}(G) = \emptyset$ .

**Theorem 2.2.** Let  $\Gamma = (G, \mathcal{C})$  be a graph with conditions. Suppose that each countable graph with finite conditions  $\Delta = (H, \mathcal{D})$ , with  $H = (W, F)$

induced by  $G$  and  $W \cap \mathcal{I}(G) = \emptyset$ , has an unfriendly partition. Then  $\Gamma$  has an  $\omega$ -unfriendly partition.

**Proof.** We write  $G = (V, E)$  and  $\mathcal{C} = (\kappa_x, \lambda_x)_{x \in V}$ . We define by induction on the ordinal  $\alpha$  two sequences  $(V_\alpha)_{\alpha \leq \delta}$  and  $(W_\alpha)_{\alpha < \delta}$  of subsets of  $V$ .

We write  $V_0 = \emptyset$  and  $W_0 = \mathcal{I}(G)$ . For each  $\alpha \geq 1$ , supposing  $V_\beta$  and  $W_\beta$  already defined for each  $\beta < \alpha$ , we write  $V_\alpha = \cup_{\beta < \alpha} W_\beta$ ; if  $V_\alpha = V$ , then we write  $\delta = \alpha$  and we do not define  $W_\alpha$ ; otherwise, we take for  $W_\alpha$  any countable subset  $X$  of  $V - V_\alpha$  such that, for each  $x \in X$ , all the neighbours of  $x$  in  $V - V_\alpha$ , or infinitely many of them, belong to  $X$ .

For each  $\alpha < \delta$ , we consider the induced graph  $H_\alpha$  defined on  $W_\alpha$ . We define by induction on  $\alpha < \delta$  a sequence of conditions  $\mathcal{D}_\alpha = (\kappa_{\alpha,x}, \lambda_{\alpha,x})_{x \in W_\alpha}$  and a map  $\pi_\alpha : W_\alpha \rightarrow \{0, 1\}$ .

We write  $\mathcal{D}_0 = (\kappa_x, \lambda_x)_{x \in W_0}$ . As  $H_0$  is a graph of infinite minimum degree, it admits an  $\omega$ -unfriendly partition  $\pi_0$  by Theorem 1.1. Then  $\pi_0$  is also an  $\omega$ -unfriendly partition of  $(H_0, \mathcal{D}_0)$ .

For  $1 \leq \alpha < \delta$ , supposing  $\pi_\beta$  and  $\mathcal{D}_\beta$  already defined for each  $\beta < \alpha$ , we write  $\kappa_{\alpha,x} = \kappa_x + |\cup_{\beta < \alpha} \{y \in W_\beta \mid \{x, y\} \in E \text{ and } \pi_\beta(y) = 0\}|$  and  $\lambda_{\alpha,x} = \lambda_x + |\cup_{\beta < \alpha} \{y \in W_\beta \mid \{x, y\} \in E \text{ and } \pi_\beta(y) = 1\}|$  for each  $x \in W_\alpha$ . It follows from Proposition 2.1 and the hypotheses of the Theorem that  $(H_\alpha, \mathcal{D}_\alpha)$  admits an  $\omega$ -unfriendly partition  $\pi_\alpha$ .

The map  $\pi = \cup_{\alpha < \delta} \pi_\alpha$  is an  $\omega$ -unfriendly partition of  $\Gamma$ . ■

Now we show that, in order to prove the existence of unfriendly partitions for countable graphs with conditions, it suffices to consider countable graphs without conditions. By Proposition 2.1, it is enough to consider the pairs  $(G, \mathcal{C})$  where  $G = (V, E)$  is a countable graph and  $\mathcal{C} = (m_x, n_x)_{x \in V}$  is a set of finite conditions with  $m_x = n_x = 0$  for each  $x$  of infinite degree.

For each such pair, we define a graph  $H = (W, F)$  as follows: We consider the union of three disjoint copies  $((W_1, F_1), \mathcal{D}_1)$ ,  $((W_2, F_2), \mathcal{D}_2)$ ,  $((W_3, F_3), \mathcal{D}_3)$  of  $(G, \mathcal{C})$ . We introduce some new vertices:

$$u_1, u_2, u_3, (v_{i,n})_{i=1,2,3;n \in \mathbb{N}}, (w_{i,n})_{i=1,2,3;n \in \mathbb{N}}, (x_{i,n})_{i=1,2,3;n \in \mathbb{N}},$$

and some new edges:

$$\begin{aligned} &\{u_i, v_{j,n}\} \text{ for } i \neq j \text{ in } \{1, 2, 3\} \text{ and } n \in \mathbb{N}; \\ &\{v_{i,3n}, w_{i,n}\}, \{v_{i,3n+1}, w_{i,n}\}, \{v_{i,3n+2}, w_{i,n}\} \text{ for } i \in \{1, 2, 3\} \text{ and } n \in \mathbb{N}; \\ &\{w_{i,2n}, x_{i,n}\}, \{w_{i,2n+1}, x_{i,n}\} \text{ for } i \in \{1, 2, 3\} \text{ and } n \in \mathbb{N}. \end{aligned}$$

For each  $i \in \{1, 2, 3\}$ , as a substitute to  $\mathcal{D}_i$ , we put  $m_y$  edges of type  $\{w_{i,n}, y\}$  and  $n_y$  edges of type  $\{x_{i,n}, y\}$  for each  $y \in W_i$ , all of them defined in such a way that each  $w_{i,n}$ , and each  $x_{i,n}$ , is an endpoint of at most one edge of that type.

Now we show that each unfriendly partition  $\pi$  of  $H$  induces an unfriendly partition of  $(G, \mathcal{C})$ . We consider  $i \neq j$  in  $\{1, 2, 3\}$  such that  $\pi(u_i) = \pi(u_j)$ .

We can suppose for instance that  $i = 2$  and  $j = 3$ . Replacing if necessary  $\pi$  by  $1 - \pi$ , we can also suppose  $\pi(u_2) = \pi(u_3) = 0$ .

Then we necessarily have  $\pi(v_{1,n}) = 1$ ,  $\pi(w_{1,n}) = 0$  and  $\pi(x_{1,n}) = 1$  for each  $n \in \mathbb{N}$ . It follows that  $\pi$  induces an unfriendly partition of  $((W_1, F_1), \mathcal{D}_1)$ , and therefore an unfriendly partition of  $(G, \mathcal{C})$ .

We observe that, in  $H$ , the only vertices of infinite degree are:  $u_1, u_2, u_3$ , which have no neighbour of infinite degree; for  $i \in \{1, 2, 3\}$ , the vertices of infinite degree of  $(W_i, F_i)$ , whose only neighbours in  $H$  are their neighbours in  $(W_i, F_i)$ . In particular,  $\mathcal{I}(G) = \emptyset$  implies  $\mathcal{I}(H) = \emptyset$ .

In view of Theorem 2.2, we have:

**Corollaire 2.3.** The following properties are equivalent:

- 1) each graph with conditions has an  $\omega$ -unfriendly partition;
- 2) each graph has an  $\omega$ -unfriendly partition;
- 3) each countable graph has an unfriendly partition;
- 4) each countable graph without nonempty induced subgraph of infinite minimum degree has an unfriendly partition.

The proposition below implies that we can consider an even smaller class of graphs in order to prove the existence of unfriendly partitions:

**Proposition 2.4.** For each graph  $G = (V, E)$ , there exists a unique partition  $V = V^1 \cup V^2 \cup \mathcal{I}(G)$  such that each element of  $V^1$  has finitely many neighbours in  $V^1$  and each element of  $V^2$  has infinitely many neighbours in  $V^1$ .

**Proof.** We define by induction on the ordinal  $\alpha$  the subsets  $V_\alpha^1, V_\alpha^2$  with  $V_\alpha^1 = \{x \in V - \cup_{\beta < \alpha} V_\beta^1 - \cup_{\beta < \alpha} V_\beta^2 \mid x \text{ has finitely many neighbours in } V - \cup_{\beta < \alpha} V_\beta^2\}$  and  $V_\alpha^2 = \{x \in V - \cup_{\beta \leq \alpha} V_\beta^1 - \cup_{\beta < \alpha} V_\beta^2 \mid x \text{ has finitely many neighbours in } V - \cup_{\beta \leq \alpha} V_\beta^1 - \cup_{\beta < \alpha} V_\beta^2\}$ . We write  $V^1 = \cup_{\gamma < \delta} V_\gamma^1$ ,  $V^2 = \cup_{\gamma < \delta} V_\gamma^2$  and  $V^3 = V - (V^1 \cup V^2)$  where  $\delta$  is the smallest integer such that  $V_\delta^1 = V_\delta^2 = \emptyset$ .

It follows from the definition of the subsets  $V_\alpha^1$  that each element of  $V^1$  has finitely many neighbours in  $V^1 \cup V^3$ .

For each  $\alpha < \delta$ , each  $x \in V_\alpha^2$  has infinitely many neighbours in  $V - \cup_{\beta < \alpha} V_\beta^2$  since it does not belong to  $V_\alpha^1$ , and therefore infinitely many neighbours in  $\cup_{\beta \leq \alpha} V_\beta^1$  since it only has finitely many neighbours in  $V - \cup_{\beta \leq \alpha} V_\beta^1 - \cup_{\beta < \alpha} V_\beta^2$ . It follows that each element of  $V^2$  has infinitely many neighbours in  $V^1$ .

The definition of  $\delta$  implies that each element of  $V^3$  has infinitely many neighbours in  $V^3$ . Consequently, we have  $V^3 \subset \mathcal{I}(G)$ .

It remains to be proved that  $\mathcal{I}(G) \subset V^3$ . We show by induction on the ordinal  $\alpha$  that  $V_\alpha^1 \cap \mathcal{I}(G) = \emptyset$  and  $V_\alpha^2 \cap \mathcal{I}(G) = \emptyset$ . If these two properties

are true for each  $\beta < \alpha$ , then we have  $V_\alpha^1 \cap \mathcal{I}(G) = \emptyset$  since each  $x \in V_\alpha^1$  has finitely many neighbours in  $V - \cup_{\beta < \alpha} V_\beta^2$ , which contains  $\mathcal{I}(G)$  by the induction hypothesis. Similarly, we have  $V_\alpha^2 \cap \mathcal{I}(G) = \emptyset$  because each  $x \in V_\alpha^2$  has finitely many neighbours in  $V - \cup_{\beta \leq \alpha} V_\beta^1 - \cup_{\beta < \alpha} V_\beta^2$ , which contains  $\mathcal{I}(G)$  by the induction hypothesis since  $V_\alpha^1 \cap \mathcal{I}(G) = \emptyset$ .

Now let us consider another partition  $V = W^1 \cup W^2 \cup \mathcal{I}(G)$  such that each element of  $W^1$  has finitely many neighbours in  $W^1$  and each element of  $W^2$  has infinitely many neighbours in  $W^1$ . Then we prove by induction on the ordinal  $\alpha$  that  $V_\alpha^1 \subset W^1$  and  $V_\alpha^2 \subset W^2$ . If these two properties are true for each  $\beta < \alpha$ , then we have  $V_\alpha^1 \subset W^1$  since each  $x \in V_\alpha^1$  has finitely many neighbours in  $V - \cup_{\beta < \alpha} V_\beta^2$ , which contains  $W^1$  by the induction hypothesis. Similarly, we have  $V_\alpha^2 \subset W^2$  because each  $x \in V_\alpha^2$  has infinitely many neighbours in  $\cup_{\beta \leq \alpha} V_\beta^1$ , which is contained in  $W^1$  by the induction hypothesis since  $V_\alpha^1 \subset W^1$ . ■

By Theorem 2.2, we can suppose that the graph  $G = (V, E)$  of Proposition 2.4 satisfies  $\mathcal{I}(G) = \emptyset$ . Then we consider the set  $V^F$  consisting of its vertices of finite degree, and the graph  $G^* = (V, E^*)$ , where  $E^*$  is obtained from  $E$  by deleting the vertices  $\{x, y\}$  for  $x, y \in V^1 - V^F$  and for  $x, y \in V^2$ . Any unfriendly partition of  $G^*$  is also an unfriendly partition of  $G$ .

Consequently, it suffices to prove the existence of unfriendly partitions for the countable graphs  $G = (V, E)$  such that  $V = V^1 \cup V^2$  and such that there exists no edge  $\{x, y\}$  for  $x, y \in V^1 - V^F$  and for  $x, y \in V^2$ .

In view of this simplification, we propose the question below. A positive answer would be a first step to prove that each countable graph has an unfriendly partition. We note that [3] does not give an answer to this question, even though it proves the existence of unfriendly partitions for large classes of countable graphs.

**Question.** Does any countable graph  $G = (V, E)$  admit an unfriendly partition if each element of  $V - V^F$  has infinitely many neighbours in  $V^F$  and no neighbour in  $V - V^F$ ?

## References

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